

**CAMBRIDGE INTERNATIONAL EXAMINATIONS**

Cambridge International Advanced Level

## **MARK SCHEME for the October/November 2015 series**

### **9231 FURTHER MATHEMATICS**

**9231/23**

Paper 2, maximum raw mark 100

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Question Number	Mark Scheme Details	Part Mark	Total
<b>1</b>	<p>Find 3 independent equations for <math>T, R_A, R_B</math>:</p> <p>Resolve horizontally: <math>R_B = T \cos \alpha</math> <b>M1 A1</b></p> <p>Resolve vertically: <math>R_A = W + T \sin \alpha</math> <b>M1 A1</b></p> <p>Take moments about <math>A</math>: (<math>a</math> may be omitted from moment eqns)  <math>R_B 3a \sin \theta = W (3a/2) \cos \theta</math>  <math>+ T a (\sin \alpha \cos \theta + \cos \alpha \sin \theta)</math>  <i>or</i> <math>+ T a \sin (\alpha + \theta)</math>  <i>or</i> <math>+ T 3a \cos \theta \sin \alpha</math> <b>M1 A1</b></p> <p>Take moments about <math>B</math>:  <math>R_A 3a \cos \theta = W (3a/2) \cos \theta</math>  <math>+ T 2a (\sin \alpha \cos \theta + \cos \alpha \sin \theta)</math>  <i>or</i> <math>+ T 2a \sin (\alpha + \theta)</math>  <i>or</i> <math>+ T 3a \sin \theta \cos \alpha</math> <b>(M1 A1)</b></p> <p>Take moments about <math>C</math>:  <math>R_A a \cos \theta + W (a/2) \cos \theta</math>  <math>= R_B 2a \sin \theta</math> <b>(M1 A1)</b></p> <p>Take moments about <math>D</math>:  <math>R_A 3a \cos \theta - W (3a/2) \cos \theta</math>  <math>= R_B 3a \sin \theta</math> <b>(M1 A1)</b></p> <p>Solve for <math>T, R_A, R_B</math> (AEF in <math>W</math> and <math>\alpha</math>):  <math>T = W / 2 \sin \alpha</math> <i>or</i> <math>\frac{1}{2}W \operatorname{cosec} \alpha</math> <b>B1</b>  <math>R_A = 3W / 2</math> <b>B1</b>  <math>R_B = W / 2 \tan \alpha</math> <i>or</i> <math>\frac{1}{2}W \cot \alpha</math> <b>B1</b></p>	9	<b>9</b>
<b>2</b>	<p>For <math>A</math> &amp; <math>B</math> use conservation of momentum, e.g.: <math>2mv_A + mv_B = 2mu</math> (allow <math>2v_A + v_B = 2u</math>) <b>M1</b></p> <p>Use Newton's law of restitution (consistent signs): <math>v_B - v_A = eu</math> <b>M1</b></p> <p>Combine to find <math>v_A</math> and <math>v_B</math>:  <math>v_A = (2 - e)u/3, v_B = 2(1 + e)u/3</math> <b>A1, A1</b></p> <p>Find <math>e</math> from <math>v_A =  v_B' </math> with <math>v_B' = [-] 0.4 v_B</math>:  <math>(2 - e) = 0.8(1 + e), e = 2/3</math> <b>M1 A1</b></p> <p><i>EITHER</i>: Equate times in terms of reqd. distance <math>x</math>: <math>(d - x)/v_A = d/v_B + x/v_B'</math> (AEF) <b>M1 A1</b>  [speeds need not be found:  <math>v_A = v_B' = 4u/9, v_B = 10u/9</math>  Use <math>v_A = v_B' = 0.4 v_B</math> to solve for <math>x</math>:  <math>d - x = 0.4 d + x, x = 0.3 d</math> <b>M1 A1</b></p> <p><i>OR</i>: Find dist. moved by <math>A</math> when <math>B</math> reaches wall:  <math>d_A = (d/v_B) v_A = 0.4 d</math> <b>(M1 A1)</b>  Find reqd. distance <math>x</math>:  <math>x = \frac{1}{2}(d - d_A) = 0.3 d</math> <b>(M1 A1)</b></p>	4	<b>10</b>

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<b>3</b>	Find $k$ by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg(a/2)/a = 2mg(3a/2 - ka)/ka$ $1/2 = 3/k - 2, \quad k = 6/5 \text{ or } 1.2$	<b>M1 A1</b> <b>A1</b>	3
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m \frac{d^2x}{dt^2} = -mg(a/2 + x)/a$ $+ 2mg(3a/2 - ka - x)/ka$ or $m \frac{d^2y}{dt^2} = +mg(a/2 - y)/a$ $- 2mg(3a/2 - ka + y)/ka$	<b>M1 A2</b>	
	Simplify to give standard SHM eqn, e.g.: <b>S.R.:</b> B1 if no derivation (max 2/5)	$\frac{d^2x}{dt^2} = - (1 + 2/k)gx/a$ $= - 8gx/3a$	<b>A1</b>	5
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$ : $T = 2\pi\sqrt{(3a/8g)}$ or $\pi\sqrt{(3a/2g)}$ ( $\sqrt$ on $\omega$ )	or $3.85\sqrt{(a/g)}$ or $1.22\sqrt{a}$ [s]	<b>B1</b> <sup>4</sup>	
	Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$ :	$0.7^2 = (8g/3a)\{(0.2a)^2 - (0.05a)^2\}$	<b>M1 A1</b>	
	Solve to find numerical value of $a$ :	$0.49 = (8g/3) \times 0.0375a, \quad a = 0.49$	<b>A1</b>	3

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4	<i>EITHER:</i> Find tension at top from $F = ma$ vertically: $T = mu^2/a - mg$ <b>B1</b>	2	13
	<i>OR:</i> Use energy at e.g. $\theta$ to upward vertical: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta)$ Find tension $T$ by using $F = ma$ radially: $T = mv^2/a - mg \cos \theta$ Eliminate $v^2$ : $= mu^2/a + mg(2 - 3 \cos \theta)$ Find $T$ at top by taking $\theta = 0$ : $T = mu^2/a - mg$ <b>(B1)</b>		
	Find $u_{\min}$ by requiring $T \geq 0$ at top [ $T > 0$ ]: $u^2/a - g \geq 0$ so $u_{\min} = \sqrt{ag}$ <b>A.G.</b> <b>B1</b>		
	Find $v$ at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ $v^2 = ag + 4ag, v = \sqrt{5ag}$ <b>M1</b> <b>A1</b>		
	Find new speed $V$ from conservation of momentum: $m'V = mv$ with $m' = m + \frac{1}{4}m$ <b>M1</b> $V = 4v/5 = 4\sqrt{ag/5}$ or $(4/5)\sqrt{5ag}$ <b>AEF</b> <b>A1</b>		
	Find $w^2$ at angle $\theta$ from conservation of energy: $\frac{1}{2}m'w^2 = \frac{1}{2}m'V^2 - m'ga(1 + \cos \theta)$ (condone $m$ instead of $m'$ here since cancels out) $[w^2 = ag(6/5 - 2 \cos \theta)]$ <b>M1</b> <b>A1</b>		
	<b>S.R.</b> Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3 \cos \theta)$ ] $\frac{1}{2}m'w^2 = \frac{1}{2}mu^2 + mga(1 - \cos \theta) - \frac{1}{4}mga(1 + \cos \theta)$ <b>(B1)</b>		
	Find tension $T'$ there by using $F = ma$ radially: $T' = m'w^2/a - m'g \cos \theta$ <b>B1</b>		
	Eliminate $w^2$ : $= m'V^2/a - m'g(2 + 3 \cos \theta)$ <b>A1</b>		
	Substitute for $m'$ and $V$ : $= (5mg/4)(6/5 - 3 \cos \theta)$ <b>AEF</b> or $3mg/2 - (15/4)mg \cos \theta$ <b>A1</b>		
Find $\cos \theta$ when string becomes slack from $T' = 0$ : $\cos \theta = \frac{1}{3} \times 6/5 = 2/5$ or 0.4 <b>M1</b> <b>A1</b> <b>S.R.</b> Allow if found from $T' = mg(6/5 - 3 \cos \theta)$			
5	Find or use sample mean <u>and</u> estimate population variance: (allow biased here: 0.412 or 0.642 <sup>2</sup> ) $\bar{x} = 222.8 / 10 = 22.28$ $s^2 = 4.12 / 9 = 0.458$ or $103/225$ or $0.677^2$ <b>M1</b>	5	5
	Find confidence interval (allow $z$ in place of $t$ ) e.g.: $22.28 \pm t\sqrt{(0.458 / 10)}$ <b>M1</b> <b>A1</b>		
	Use of correct tabular value: $t_{9, 0.975} = 2.26[2]$ <b>A1</b>		
	Evaluate C.I. correct to 3 s.f.: $22.3 \pm 0.48[4]$ or $[21.8, 22.8]$ <b>A1</b>		

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<b>6</b>	Find prob. $p$ of head from mean = $2 \times$ variance: $1/p = 2 \times (1-p)/p^2$ , $p = 2/3$ <b>A.G.</b> <b>M1 A1</b>	2	<b>8</b>
<b>(i)</b>	Find $P(X=4)$ (denoting $1-p$ by $q$ [ $= 1/3$ ]): $P(X=4) = q^3 \times p$ $= 2/81$ or 0.0247 <b>B1</b>	1	
<b>(ii)</b>	Find or state $P(X > 4)$ : $P(X > 4) [= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)] = q^4$ $= 1/81$ or 0.0123 <b>M1 A1</b>	2	
<b>(iii)</b>	Formulate condition for $N$ : Take logs (any base) to give bound for $N$ : Find $N_{\min}$ : ( $N < 6.29$ or $N = 6.29$ earns M2 A0) $1 - q^N > 0.999$ , $[(1/3)^N < 0.001]$ <b>M1</b> $N > \log 0.001 / \log 1/3$ <b>M1</b> $N > 6.29$ , $N_{\min} = 7$ <b>A1</b>	3	
<b>7</b>	Find $F(x)$ for $1 \leq x \leq 4$ : $F(x) = (x^3 - 1)/63$ <b>B1</b>	5	
	Find $G(y)$ from $Y = X^2$ for $1 \leq x \leq 4$ : (result may be stated) $G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$ <b>M1 A1</b>		
	Find $g(y)$ for corresponding range of $y$ : $g(y) = y^{1/2}/42$ <b>A.G.</b> <b>A1</b>		
	Find or state corresponding range of $y$ : $1 \leq y \leq 16$ <b>A.G.</b> <b>B1</b>		
<b>(i)</b>	Find median value $m$ of $Y$ : $(m^{3/2} - 1)/63 = 1/2$ $m = 32.5^{2/3} = 10.2$ <b>M1 A1</b>	2	
<b>(ii)</b>	Find $E(Y)$ [or equivalently $E(X^2)$ ]: $E(Y) = \int y g(y) dy = \int y^{3/2} dy / 42$ $= [y^{5/2}]_1^{16} / 105 = 1023/105$ $= 341/35$ or 9.74 <b>M1 A1</b>	2	<b>9</b>

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<b>8</b>	<p>Find mean of sample data [for use in Poisson distn.]:  <math>\lambda = 220/100 = 2.2</math> <b>B1</b></p> <p>State (at least) null hypothesis (AEF):  <math>H_0</math>: Poisson distn. fits data  or <math>\lambda = 2.2</math> <b>B1</b></p> <p>Find expected values <math>100\lambda^r e^{-\lambda}/r!</math> (to 1 d.p.):  (ignore incorrect final value here for M1)  11.080 24.377 26.814 19.664  10.8151 4.759 2.491 <b>M1 A1</b></p> <p>Combine last two cells so that exp. value <math>\geq 5</math>:  <math>O_i</math>: 3  <math>E_i</math>: 7.25 <b>M1*</b></p> <p>Calculate value of <math>\chi^2</math> (to 2 d.p.; A1 dep M1*):  (allow 7.95 if 1 d.p. exp.values used)  <math>\chi^2 = 0.076 + 2.879 + 0.653 + 1.448</math>  <math>+ 0.441 + 2.491</math>  <math>= 7.99</math> <b>M1 A1</b></p> <p>State or use consistent tabular value (to 3 s.f.):  5 cells: <math>\chi_{3,0.95}^2 = 7.815</math>  6 cells: <math>\chi_{4,0.95}^2 = 9.488</math> (correct)  7 cells: <math>\chi_{5,0.95}^2 = 11.07</math> <b>B1</b></p> <p>State or imply valid method for conclusion e.g.:  Accept <math>H_0</math> if <math>\chi^2 &lt;</math> tabular value <b>M1</b></p> <p>Conclusion (AEF, requires both values correct):  Distn fits or <math>\lambda = 2.2</math> <b>A1</b>  Not combining cells [so <math>\chi^2 = 8.64</math>] can earn  B1 B1 M1 A1 M0 M1 B1 M1 (max 7)</p>	10	<b>10</b>
<b>9</b>	<p>Calculate gradient <math>b_1</math> in <math>y - \bar{y} = b_1(x - \bar{x})</math>:  <math>S_{xy} = 24\,879 - 472 \times 400/8</math>  <math>= 1\,279</math>  <math>S_{xx} = 29\,950 - 472^2/8 = 2\,102</math>  <math>b_1 = S_{xy} / S_{xx} = 0.6085</math> (3 s.f.) <b>M1 A1</b></p> <p>Find regression line of <math>y</math> on <math>x</math>:  <math>y = 400/8 + b_1(x - 472/8)</math> <b>M1 A1</b>  <math>= 50 + 0.6085(x - 59)</math>  <math>= 0.6085x + 14.1</math></p> <p>Find <math>y</math> when <math>x = 72</math>:  <math>= 57.9</math> or <math>58</math></p> <p>Allow use of regression line of <math>x</math> on <math>y</math>  (since neither variable clearly independent):  <math>S_{yy} = 21\,226 - 400^2/8 = 1\,226</math>  <math>b_2 = S_{xy} / S_{yy} = 1.043</math> <b>(M1 A1)</b>  <math>x = 472/8 + b_2(y - 400/8)</math> <b>(M1 A1)</b>  <math>= 1.043y + 6.85</math>  <math>Y = 62.5</math> or <math>62</math> <b>(A1)</b></p>	5	

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	Find product moment correlation coefficient $r$ : $r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ <i>or</i> $\sqrt{(0.6085 \times 1.043)} = 0.797$ <b>M1 A1*</b> State both hypotheses (B0 for $r \dots$ ): $H_0: \rho = 0, H_1: \rho \neq 0$ <b>B1</b> State or use correct tabular two-tail $r$ -value: $r_{8,5\%} = 0.707$ <b>B1*</b> State or imply valid method for conclusion e.g.: Reject $H_0$ if $ r  >$ tab. value (AEF) <b>M1</b> Correct conclusion (AEF, dep A1*, B1*): There is non-zero correlation <b>A1</b>	6	<b>11</b>
<b>10A</b>	Find MI of lamina about $Q$ : $I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2$ <b>M1 A1</b> $[= (15/4 + 81/4) ma^2 = 24 ma^2]$ State or find MI of rod about $Q$ : $I_{\text{rod}} = (\frac{1}{3} + 1) M(3a/2)^2$ $[= 3Ma^2]$ <b>B1</b> Sum to find MI of object about $Q$ : $I_1 = 24 ma^2 + 3 Ma^2$ $= 3(8m + M) a^2$ <b>A.G.</b> <b>A1</b>	4	
	Find MI of object about mid-point of $PQ$ : $I_2 = (15/4 + 3^2) ma^2 + \frac{1}{3} M(3a/2)^2$ $= (51/4) ma^2 + \frac{3}{4} Ma^2$ $= \frac{3}{4} (17m + M) a^2$ <b>A.G.</b> <b>M1 A1</b>	2	
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis $l_1$ : $[-]I_1 d^2\theta/dt^2 = mg \times (9a/2) \sin \theta + Mg \times (3a/2) \sin \theta$ <b>M1 A1</b> $[= (9m/2 + 3M/2) ga \sin \theta]$ [Approximate $\sin \theta$ by $\theta$ and] find $\omega_1^2$ in SHM eqn: $\omega_1^2 = (3m + M)g / 2(8m + M) a$ <b>M1</b>		
	Find period $T_1$ for axis $l_1$ from $2\pi/\omega_1$ : (AEF) $T_1 = 2\pi\sqrt{\{2(8m + M) a / (3m + M)g\}}$ <b>A1</b>		
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis $l_2$ : $[-]I_2 d^2\theta/dt^2 = mg \times 3a \sin \theta$ <b>M1</b> [Approximate $\sin \theta$ by $\theta$ and] find $\omega_2^2$ in SHM eqn: $\omega_2^2 = 4mg / (17m + M) a$ <b>M1</b>		
	Find period $T_2$ for axis $l_2$ from $2\pi/\omega_2$ : (AEF) $T_2 = 2\pi\sqrt{\{(17m + M) a / 4mg\}}$ <b>A1</b>		
	Verify that $T_1 = T_2$ when $m = M$ : (AEF) $T_1 = 2\pi\sqrt{(18 a / 4g)} = T_2$ <b>B1</b> [Taking $m = M$ throughout 2 <sup>nd</sup> part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]	8	<b>14</b>

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<b>10B</b>	State hypotheses (B0 for $\bar{x}$ ...), e.g.: $H_0: \mu_X = \mu_Y, H_1: \mu_X \neq \mu_Y$	<b>B1</b>	9	
	State assumption (AEF): Distributions have equal variances	<b>B1</b>		
	Find sample means <u>and</u> estimate popln. variances: $\bar{x} = 4.2, \bar{y} = 4.8$ $s_X^2 = (180 - 42^2/10) / 9$ (allow biased here: 0.36 or 0.6 <sup>2</sup> ) $= 0.4$ or 0.6325 <sup>2</sup> $s_Y^2 = (281.5 - 57.6^2/12) / 11$ (allow biased here: 0.4183 or 0.6468 <sup>2</sup> ) $= 0.4564$ or 251/550 or 0.6755 <sup>2</sup>	<b>M1</b>		
	Estimate (pooled) common variance: (note $s_X^2$ and $s_Y^2$ not needed explicitly)	$s^2 = (9 s_X^2 + 11 s_Y^2) / 20$ (AEF) or $(180 - 42^2/10 + 281.5 - 57.6^2/12) / 20$ $= 0.431$ or 0.6565 <sup>2</sup>		<b>M1 A1</b>
	Calculate value of $t$ (to 3 s.f.):	$[-] t = (\bar{y} - \bar{x}) / s \sqrt{(1/10 + 1/12)}$ $= 2.13$		<b>M1 A1</b>
	State or use correct tabular $t$ value: (or can compare $\bar{y} - \bar{x} = 0.6$ with 0.586)	$t_{20, 0.975} = 2.086$ [allow 2.09]		<b>B1*</b>
	Correct conclusion (AEF, $\sqrt$ on $t$ , dep *B1):	$t > 2.09$ so mean masses not same		<b>B1<sup>✓</sup></b>
	<b>S.R.</b> Implicitly taking $s_X^2, s_Y^2$ as popln. variances: (may also earn first B1 B1 M1)	$z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0.6 / \sqrt{(0.078)} = 2.15$		
	Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9)	$2.15 > 1.96$ so mean masses not same		<b>(B1)</b>
	State hypotheses (B0 for $\bar{x}$ ...), e.g.:	$H_0: \mu_X = 3.8, H_1: \mu_X > 3.8$ or $H_0: \mu_X = \mu_Z, H_1: \mu_X > \mu_Z$		<b>B1</b>
	Calculate value of $t$ using $s_X$ from above:	$t = (4.2 - 3.8) / (s_X/\sqrt{10}) = 2.0$		<b>M1 A1</b>
	State or use correct tabular $t$ value: (or can compare 0.4 with 0.367)	$t_{9, 0.95} = 1.833$ [allow 1.83]		<b>B1*</b>
	Correct conclusion (A.E.F., $\sqrt$ on $t$ , dep *B1):	$t > 1.833$ , so claim is justified or mean mass of Royals > mean mass of Crowns		<b>B1<sup>✓</sup></b>
			<b>14</b>	